

# ON THE STRUCTURE OF DEMAILLY-SEMPLÉ INVARIANT JET DIFFERENTIALS

JINGZHOU SUN

ABSTRACT.

In the terminology of [De97], a directed manifold is a pair  $(X, V)$ , where  $X$  is a complex manifold and  $V \subset T_X$  a subbundle. Let  $(X, V)$  be a complex directed manifold,  $J_k V \rightarrow X$  is defined to be the bundle of  $k$ -jets of germs of curves  $f : (\mathbb{C}, 0) \rightarrow X$  which are tangent to  $V$ , i.e., such that  $f'(t) \in V_{f(t)}$  for all  $t$  in a neighborhood of 0, together with the projection map  $f \rightarrow f(0)$  onto  $X$ . It is easy to check that  $J_k V$  is actually a subbundle of  $J_k T_X$ . Let  $\mathbb{G}_k$  be the group of germs of  $k$ -jet biholomorphisms of  $(\mathbb{C}, 0)$ , that is, the group of germs of biholomorphic maps

$$t \rightarrow \varphi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k, \quad a_1 \in \mathbb{C}^*, a_j \in \mathbb{C}, \quad j > 2$$

in which the composition law is taken modulo terms  $t^j$  of degree  $j > k$ . The group  $\mathbb{G}_k$  acts on the left on  $J_k V$  by reparametrization,  $(\varphi, f) \rightarrow f \circ \varphi$ .

Given a directed manifold  $(X, V)$  with  $\text{rank } V = r$ , let  $\tilde{X} = \mathbb{P}(V)$ . The subbundle  $\tilde{V} \subset T_{\tilde{X}}$  is defined by

$$\tilde{V}_{x,[v]} = \{\xi \in T_{\tilde{X},(x,[v])} \mid \pi_* \xi \in \mathbb{C} \cdot v\}$$

for any  $x \in X$  and any  $v \in T_{X,x} \setminus \{0\}$ . Starting with a directed manifold  $(X, V) = (X_0, V_0)$ , we get a tower of directed manifolds  $(X_k, V_k)$ , called Demailly-Semple  $k$ -jet bundle of  $X$ , defined by  $X_k = \tilde{X}_{k-1}$ ,  $V_k = \tilde{V}_{k-1}$ . In particular, when  $X$  is a hypersurface in  $\mathbb{CP}^3$ , we start with  $(X, T_X)$

The line bundle  $\mathcal{O}_{X_k}(1)$  will be called the Demailly-Semple  $k$ -jet line bundle.

**Theorem 0.1.** [De97] *The direct image sheaf  $(\pi_{k,0})_* \mathcal{O}_{X_k}(m)$  on  $X$  coincides with the (locally free) sheaf  $E_{k,m} V^*$  of  $k$ -jet differentials of weighted degree  $m$ , that is, by definition, the set of germs of polynomial differential operators*

$$Q(f) = \sum_{\alpha_1 \cdots \alpha_k} a_{\alpha_1 \cdots \alpha_k}(f) (f')^{\alpha_1} (f'')^{\alpha_2} \cdots (f^{(k)})^{\alpha_k} \quad (1)$$

on  $J_k V$  (in multi-index notation,  $(f')^{\alpha_1} = ((f'_1)^{\alpha_{1,1}} (f'_2)^{\alpha_{1,2}} \cdots (f'_r)^{\alpha_{1,r}})$ , which are moreover invariant under arbitrary changes of parametrization: a germ of operator  $Q \in E_{k,m} V^*$  is characterized by the condition that, for every germ  $f \in J_k V$  and every germ  $\varphi \in \mathbb{G}_k$ ,

$$Q(f \circ \varphi) = \varphi^{!m} Q(f) \circ \varphi$$

Given a finite dimensional vector space  $W$ , we can define  $E_{k,m} W^*$ . Seeing that  $W$  can be seen as the tangent space of its own, we define  $E_{k,m} W^*$  to be the fibre at the origin of the bundle  $E_{k,m} T_W^*$ . More generally, if we have a vector bundle  $V$  of finite rank over a manifold

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$X$ . Let  $G$  be the structure group of  $V$ , and let  $\mathcal{P}$  be the principle bundle of  $V$ . We denote a fibre by  $W$ . An element  $g \in G$  acts on  $W$ , inducing an automorphism of  $E_{k,m}W^*$ . Therefore, we get a representation of  $G$  in  $E_{k,m}W^*$ . We can define

$$\tilde{E}_{k,m}V^* \triangleq \mathcal{P} \times_G E_{k,m}W^*$$

However when  $V$  is a subbundle of  $T_X$ , this vector bundle does NOT coincide with the original definition of  $E_{k,m}V^*$ . One can have a sense of this by taking a collection of covering charts of  $X$  such that  $V$  is trivial on each open set. Consider  $V_1$  as a fiber bundle over  $X$  with each fiber a vector bundle of rank  $r$  over  $\mathbb{CP}^{r-1}$ . One can then compute the transition functions of  $V_1$  on overlap of intersecting charts, and see that the second derivatives of the transition functions of  $X$  are involved. The following example shows  $E_{k,m}V^*$  is indeed not a representation of the principle bundle of  $V$ .

Suppose  $X$  is a surface, and  $V = T_X$ . In a neighborhood of a point  $x \in X$  such that  $T_X$  is trivial, one can easily compute that  $E_{2,3}T_X^*|_x$  as a representation of  $GL(T_{X,x})$  decomposes as a direct sum  $S^3T_{X,x}^* \oplus K_{X,x}$ . If  $E_{2,3}T_X^* = \mathcal{P} \times_{GLT_{X,x}} E_{2,3}T_X^*|_x$ , this decomposition will globalize to give a decomposition  $E_{2,3}T_X^* = S^3T_X^* \oplus K_X$ . But then we will have  $H^0(X, E_{2,3}T_X^* \otimes (-K_X)) \neq 0$

Recall that in [DeE]  $\theta_{k,m}$  is defined to be the smallest rational number such that  $H^0(X, E_{k,m}T_X^* \otimes (tK_X)) \neq 0$ , assuming  $tK_X$  is an integral divisor,  $t \in \mathbb{Q}$ . Under our assumption, we get that  $\theta_{2,3} \leq -\frac{1}{3}$ . On the other hand, the following inequality was proved in [DeE]

**Proposition 0.2.** [DeE] *Let  $X$  be a generic surface of degree  $d \geq 6$  in  $\mathbb{CP}^3$ . Then*

$$\theta_{2,m} \geq \frac{-1}{2m} + \frac{2 - 7/(2m)}{d - 4}, \quad \text{for } m = 3, 4, 5$$

We get a contradiction.

This example gives us the impression that the global structure of  $E_{k,m}T_X^*$  is complicated. Therefore, knowing the local structure of  $E_{k,m}T_X^*$  does not enable one to get global conclusions before one understand the transition functions for  $E_{k,m}T_X^*$  on overlaps of open covering of  $X$ .

## REFERENCES

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DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218, USA

*E-mail address:* jzsun@math.jhu.edu